Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

CHAPTER III.

In determining the equations of lowest form upon which depends the inscription of a regular polygon, it might appear that we can employ a random grouping of the chords. But this grouping is *not* arbitrary but according to a profound principle of the Theory of Numbers, the exposition of which is beyond the scope of this paper.

For example, the six chords of the regular 13-gon were formed into two groups, $(A_1+A_3-A_4)$ and $(A_5-A_2-A_6)$. By any different grouping, the product of the two groups could not be expressed rationally in terms of the unit radius, but would involve the chords themselves.

The ordinary rule may be stated thus: Let n be the number of sides of the regular polygon and write $p=\dfrac{n-1}{2}$. The subscripts are written in the definite order: 1, its double 2, its double 4, etc, until a number $> p$ is obtained when we take the difference between it and n, take its double, etc, at last obtaining the number p.

Thus for $n=13$, we get the following *order* of the subscripts: 1,2,4,5,3,6.

To group them into *two* groups, choose every second subscript, thus: 1, 4, 3 and 2, 5, 6.

To group them into *three* groups, choose everp third subscript, thus: 1, 5; also 2, 3; also 4, 6.

In some rare cases (as that of the regular 257-gon) we must treble the subscript each time instead of doubling it to obtain the required *order*.

Let us consider the regular polygon of 17 *side.*

It was with rare foresight that the youthful discoverer Gauss declared to his room-mate that the discovery of its geometric inscriptibility was alone sufficient to make his name immortal.

The sum of the two groups $(A_1-A_2-A_4-A_8)$ and $(A_3+A_5-A_6+A_7)$ equals 1. Their product expanded and reduced equals $-4(A_1-A_2+A_3-A_4+A_5-A_6+A_7-A_8)=-4$. Hence, the two groups are the roots of the quadratic $x^2-x-4=0$.

Hence, $A_1-A_2-A_4-A_8=\frac{1}{2}(1-\sqrt{17})$; $A_3+A_5-A_6+A_7=\frac{1}{2}(1+\sqrt{17})$.

Forming the sub-groups, (A_1-A_4) and $(-A_2-A_8)$, their product $=-(A_1-A_2+A_3-A_4+A_5-A_6+A_7-A_8)=-1$.

Similarily, $(A_3+A_5)(-A_6+A_7)=-1$.

Hence, (A_1-A_4) and $(-A_2-A_8)$ are the roots of $x^2-\frac{1}{2}(1-\sqrt{17})x-1=0$. Also (A_3+A_5) and $(-A_6+A_7)$ are the roots of $x^2-\frac{1}{2}(1+\sqrt{17})x-1=0$.

$\therefore A_1-A_4=\frac{1}{4}[1-\sqrt{17}+\sqrt{2(17-\sqrt{17})}]$; $-A_2-A_8=\frac{1}{4}[1-\sqrt{17}-\sqrt{2(17-\sqrt{17})}]$

$A_3+A_5=\frac14[1+\sqrt{17}+\sqrt{2(17+\sqrt{17})}]$; $-A_6+A_7=\frac14[1+\sqrt{17}-\sqrt{2(17+\sqrt{17})}]$

Now $A_1.A_4=A_3+A_5$; $A_2.A_8=A_6-A_7$

$A_3.A_5=A_2+A_8$; $A_6.A_7=A_1-A_4$.

Hence, A_1 and $-A_4$ are the roots of the quadratic

$$x^2-\frac14[1-\sqrt{17}+\sqrt{2(17-\sqrt{17})}]x-\frac14[1+\sqrt{17}+\sqrt{2(17+\sqrt{17})}]=0.$$

Thus $A_1=\frac18[1-\sqrt{17}+\sqrt{[2(17-\sqrt{17})]}$

$$+\sqrt{68+12\sqrt{17}+16\sqrt{[2(17+\sqrt{17})]}+2(1-\sqrt{17})\sqrt{[2(17-\sqrt{17})]}}].$$

Similarly for the other seven chords.

In a later chapter will be given a beautiful Geometric construction for inscribing the regular 17-gon founded upon the method invented by Von Staudt for graphically finding the roots of any quadratic.

For the regular 19-*gon*, the order of the subscripts is 1,2,4,8,3,6,7,5,9. Hence the groups are $(A_1-A_8+A_7),(-A_2+A_3+A_5)$, and $(-A_4-A_6+A_9)$.

$(A_1-A_8+A_7)(-A_2+A_3+A_5)$, expanded gives $(-A_1-2A_2+2A_3-3A_4+2A_5-3A_6+A_7-A_8+3A_9)=-[1+(-A_2+A_3+A_5)+2(-A_4-A_6+A_9)]$.

Similarly, $(A_1-A_8+A_7)(-A_4-A_6+A_9)=-[1+(A_1-A_8+A_7)+2(-A_2+A_3+A_5)]$ $(-A_2+A_3+A_5)(-A_4-A_6+A_9)$

$$=-[1+(-A_4-A_6+A_9)+2(A_1-A_8+A_7)].$$

Write $A=(A_1-A_8+A_7)$; $B=(-A_2+A_3+A_5)$; $C=(-A_4-A_6+A_9)$ $\therefore$ $A+B+C=1$. $AB=-(1+B+2C)$; $AC=-(1+A+2b)$; $BC=-(1+C+2A)$. $\therefore$ $AB+AC+BC=-[3+3(A+B+C)]=-6$. $ABC=-A(1+C+2A)=-A(3-2B-C)=-(3A-2AB-AC)=-(3A+2+2B+4C+1+A+2B)=-[3+4(A+B+C)]=-7$. Hence, A, B, C are the three roots of the cubic $x^3-x^2-6x+7=0$. Now $(A_1.A_7-A_1.A_8-A_7.A_8)=-[(A_1-A_8+A_7)+(-A_4-A_6+A_9)]=-(A+C)=B-1$. $A_1.A_7.A_8=A_1(A_1-A_4)=2+(A_2-A_3-A_5)=2-B$. Hence, $A_1,-A_8$, and A_7 are the roots of $x^3-Ax^2+(B-1)x+(2-B)=0$. Similarly, $-A_2,A_3$, and A_5 are the roots of $x^3-Bx^2+(C-1)x+(2--C)=0$. Similarly, $-A_4,-A_6$, and A_9 are the roots of $x^3-Cx^2-(A-1)x+(2-A)=0$.

For the regular 31-gon, we find that $(A_1-A_2-A_4-A_8+A_{15})$, $(A_3-A_6+A_7-A_{12}-A_{14})$, and $(A_5+A_9-A_{10}+A_{11}+A_{13})$ are the three roots, say A, B, C, of $x^3-x^2-10x+8=0$. Further, that $A_1,-A_2,-A_4,-A_8,+A_{15}$ are the five roots of $x^5-Ax^4-(1+B)x^3-(1+2B+3C)x^2-2(1+C)x+1=0$.

That $A_3,-A_6,+A_7,-A_{12},-A_{14}$ are the roots of $x^5-Bx^4-(1+C)x^3-(1+2C+3A)x^2-2(1+A)x+1=0$. That $A_5,A_9,-A_{10},A_{11},A_{13}$ are the roots of $x^5-Cx^4-(1+A)x^3-(1+2A+3B)x^2-2(1+B)x+1=0$.

The above examples will serve to indicate the spirit of my method. In a lengthy memoir I have given a general and exhaustive treatment of the subject based on these and similar geometric principles and upon the Theory of of Numbers. It is proved that, for a regular polygon of n sides, n being a prime number, to be *geometrically* inscriptible, n must be of the form 2^m+1. For example, those of 3, 5, 17, 257, 65537, etc, sides.